SUGGESTED SOLUTIONS TO HOMEWORK I

Solution 1. (1) Since the characteristic system gives

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{xe^{-u}},$$

therefore by Lagrange theorem, the general solution is determined by

$$G\left(\frac{x}{y}, x - e^u\right) = 0,$$

where $G \in C^1$ is an arbitrary function.

(2) Since the characteristic system gives

$$\frac{dx}{x(y^2+u)} = \frac{dy}{y(x^2+u)} = \frac{du}{u(x^2-y^2)},$$

therefore by Lagrange theorem, the general solution is determined by

$$G(xyu, x^2+y^2-2u) = 0,$$

where $G \in C^1$ is an arbitrary function.

Solution 2. (1) We parametrise the curve carrying the initial data by

$$x(0,s) = 0, \quad y(0,s) = s, \quad u(0,s) = \cos s.$$

Since the characteristic equations are given by

$$\begin{aligned} \frac{dx(t,s)}{dt} &= 1, \\ \frac{dy(t,s)}{dt} &= x(t,s), \\ \frac{du(t,s)}{dt} &= y(t,s), \end{aligned}$$

therefore

$$x(t,s) = t, \quad y(t,s) = \frac{t^2}{2} + s, \quad u(t,s) = \frac{t^3}{6} + ts + \cos s,$$

solving t, s in terms of x, y, the solution is obtained as

$$u(x,y) = xy - \frac{1}{3}x^3 + \cos\left(y - \frac{x^2}{2}\right).$$

(2) We parametrise the curve carrying the initial data by

$$x(0,s) = s, \quad y(0,s) = 1, \quad u(0,s) = 2s.$$

Since the characteristic equations are given by

$$\frac{dx(t,s)}{dt} = u(t,s),$$
$$\frac{dy(t,s)}{dt} = y(t,s),$$
$$\frac{du(t,s)}{dt} = x(t,s),$$

therefore

$$x(t,s) = \frac{3}{2}se^t - \frac{1}{2}se^{-t}, \quad y(t,s) = e^t, \quad u(t,s) = \frac{3}{2}se^t + \frac{1}{2}se^{-t},$$

solving t, s in terms of x, y, the solution is obtained as

$$u(x,y) = x\frac{3y^2+1}{3y^2-1}, \quad y > \frac{\sqrt{3}}{3}.$$

Solution 3. (1) From the equation, since the characteristic system gives

$$\frac{dx}{1} = \frac{dy}{x},$$

therefore the solution is constant along the characteristic curve $\Gamma : y - \frac{x^2}{2} = c$, where $c \in \mathbb{R}$ is an arbitrary constant. Note that Γ intersects the x-axis at the points $(\pm\sqrt{-2c}, 0)$ for $c \leq 0$, therefore, it is necessary to require f to be an even function for a solution to exist.

(2) Since the characteristic curve Γ intersects the x-axis only for $c \leq 0$, therefore the solution is uniquely determined by the initial condition for $\{(x,y) : y - \frac{x^2}{2} \leq 0\}$.

Solution 4. (1) Since

$$\frac{df}{ds}(s, x_0 + tv, v) = (\partial_t f + v \cdot \nabla_x f)(s, x_0 + sv, v) = 0,$$

therefore

$$f(t,x,v) = f^0(x - sv, v).$$

Moreover, since $f^0 \in C^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)$, we have $f \in C^1(\mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$.

(2) For $p = \infty$,

$$\|f(t)\|_{L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_v)} = \|f^0\|_{L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_v)}.$$

For $1 \leq p < \infty$, by change of variables $y = x - tv$,

$$\begin{aligned}\|f(t)\|_{L^p(\mathbb{R}^3_x \times \mathbb{R}^3_v)} &= \int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v} |f^0(x - sv, v)|^p dvdx \\ &= \int_{\mathbb{R}^3_y}\int_{\mathbb{R}^3_v} |f^0(y, v)|^p dvdy \\ &= \|f^0\|_{L^p(\mathbb{R}^3_x \times \mathbb{R}^3_v)}.\end{aligned}$$

(3) Since

$$\begin{aligned}\left|\int_{\mathbb{R}^3_v} f(t,x,v)dv\right| &\leq \int_{\mathbb{R}^3_v} |f^0(x-tv,v)|dv \\ &\leq \int_{\mathbb{R}^3_v} \sup_{w\in\mathbb{R}^3} |f^0(x-tv,w)|dv \\ &\leq \frac{1}{t^3}\int_{\mathbb{R}^3_v} \sup_{w\in\mathbb{R}^3} |f^0(v')|dv'.\end{aligned}$$